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LIFTINGS FOR HAAR MEASURE ON $\{0,1\}^\kappa$

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ABSTRACT

We call $E \subseteq \{0,1\}^\kappa$ *projective* if for some countable $A \subseteq \kappa$ there is an $E_A \subseteq \{0,1\}^A$ such that $E = E_A \times \{0,1\}^{\kappa \setminus A}$ and E_A is a projective subset of the Cantor set $\{0,1\}^A$. We construct a model where Haar measure on $\{0,1\}^\kappa$ has no projective lifting (and in particular no Baire lifting) for any $\kappa \geq \omega$.

§1. Introduction

The von Neumann–Maharam lifting theorem states that if (X,Σ,μ) is a complete probability space, then there is a *lifting* ρ for μ, i.e., a Boolean homomorphism $\rho : \Sigma \to \Sigma$ such that $\rho(E) = \rho(F)$ whenever $\mu(E \Delta F) = 0$, and $\mu(\rho(E)\Delta E) = 0$ for all $E, F \in \Sigma$ (see [M], or [F, Theorem 4.4]). The following consistency result of Mokobodzki is relevant if (X,Σ,μ) is not complete.

THEOREM [F, Theorem 4.6]. *If $2^{\aleph_0} = \aleph_1$ and (X,Σ,μ) is any probability space with $|\Sigma| \leq \aleph_2$, then there is a lifting for μ.*

On the other hand, Shelah has shown in [S] that $(2^\omega, \mathcal{B}_\omega, \mu_\omega)$ (the usual Borel measure on 2^ω: see notation in §2) consistently does not have a lifting. We generalize this result by showing that consistently $(2^\kappa, \Sigma_\kappa, \mu_\kappa)$ has no lifting for any κ,

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where Σ_κ is the collection of measurable projective subsets of $\{0,1\}^\kappa$, as defined in the abstract and in §2.

§2. Notation

(a) $\mathrm{Fn}(A,B) = \{p : p$ is a function, $\mathrm{dom}(p)$ is a finite subset of A, $\mathrm{ran}(p) \subseteq B\}$.

(b) For $s \in \mathrm{Fn}(A,2)$, let $[s]_A = \{g \in 2^A : s \subseteq g\}$.

(c) $\mathcal{B}_A$ is the σ-algebra of subsets of 2^A generated by the sets $[s]_A$, where $s \in \mathrm{Fn}(A,2)$ (i.e., $\mathcal{B}_A$ consists of the *Baire subsets* of 2^A).

(d) μ_A is the unique probability measure on 2^A such that $\mathrm{dom}(\mu_A) = \mathcal{B}_A$ and $\mu_A([s]_A) = 2^{-|s|}$ for $s \in \mathrm{Fn}(A,2)$. The completion of μ_A is denoted by $\hat{\mu}_A$, its domain by $\hat{\mathcal{B}}_A$.

(e) When $A \subseteq B$, let $\pi_A^B : 2^B \to 2^A$ be the projection map given by $\pi_A^B(g) = g \restriction A$ for $g \in 2^B$. Let $\mathcal{B}_A^B = \{(\pi_A^B)^{-1}(E) : E \in \mathcal{B}_A\}$, i.e., $\mathcal{B}_A^B$ is the σ-algebra of subsets of 2^B generated by the sets $[s]_B$, for $s \in \mathrm{Fn}(A,2)$.

(f) For each infinite set A and each one-to-one map $g : \omega \to A$, define $g^* : 2^A \to 2^\omega$ by $g^*(h) = h \circ g$. Also let $\bar{g} : P(2^\omega) \to P(2^A)$ be given by $\bar{g}(E) = (g^*)^{-1}(E)$. Let Σ_ω be the algebra of measurable projective subsets of 2^ω (see [Je] for the basic facts about projective sets). Let $\Sigma_A = \{\bar{g}(E) : (E \in \Sigma_\omega$ and $g : \omega \to A$ is one-to-one$\}$, which we shall call the algebra of *measurable projective subsets* of 2^A. Let $\Sigma_A^B = \{(\pi_A^B)^{-1}(E) : E \in \Sigma_A\}$.

(g) If $\mathcal{B}$ is a subalgebra of $\mathcal{B}_A$ then $\rho : \mathcal{B} \to \hat{\mathcal{B}}_A$ is a *lifting for* μ_A *with respect to* $\mathcal{B}$ (or simply a *lifting for* μ_A if $\mathcal{B} = \mathcal{B}_A$) if

(i) ρ is a Boolean homomorphism,

(ii) $\forall E \in \mathcal{B}$, $\mu_A(E \Delta \rho(E)) = 0$,

(iii) $(\forall E,F \in \mathcal{B})(\mu_A(E\Delta F) = 0 \Rightarrow \rho(E) = \rho(F))$.

If $\rho''\mathcal{B} \subseteq \Sigma_A$, then we call ρ a *projective* lifting for μ_A with respect to $\mathcal{B}$.

(h) We code elements of Σ_ω by subsets of ω (any reasonable coding method will do), and let $\mathcal{C}_\omega$ denote the set of such codes. For $c \in \mathcal{C}_\omega$, we write $\mathrm{eval}(c)$ for the corresponding element of Σ_ω.

§3. Innocuous iterations

We build our model using a forcing technique called innocuous iteration which was developed by the second author in [Ju], to which we refer the reader for a detailed presentation of the technique. To make this paper as self-contained as possible, we reproduce here the basic properties of innocuous iterations.

3.1. Definitions. (i) If $\mathbf{P}$ is a partial order, $\mathbf{P}_0 \subseteq \mathbf{P}$ and $\mathcal{D}$ is any class, then $\mathbf{P}_0 \ll_{\mathcal{D}} \mathbf{P}$ means that

(1) $\forall p, q \in \mathbf{P}_0$, $p \perp q$ in $\mathbf{P}_0 \Rightarrow p \perp q$ in $\mathbf{P}$, and

(2) $\forall D \subseteq \mathbf{P}_0$, if D is predense in $\mathbf{P}_0$ and $D \in \mathfrak{D}$ then D is predense in $\mathbf{P}$.

(Thus $\mathbf{P}_0 \ll_V \mathbf{P}$ means that $\mathbf{P}_0$ is completely embedded in $\mathbf{P}$.)

(ii) A partial order $\mathbf{P}$ is *harmless* if $\mathbf{P}$ satisfies the c.c.c. and for every countable $\mathbf{P}_0 \subseteq \mathbf{P}$ there is a countable $\mathbf{P}_1$ such that $\mathbf{P}_0 \subseteq \mathbf{P}_1 \ll_V \mathbf{P}$.

(iii) Let $\kappa = \omega_2$. (The definition of innocuous iterations of length $\kappa > \omega_2$ is more complicated: see [Ju 2.3] and also 5.7 below.) Let $\langle\langle \mathbf{P}_\alpha \rangle_{\alpha \le \kappa}, \langle \dot{\mathbf{Q}}_\alpha \rangle_{\alpha < \kappa}\rangle$ be a finite support κ-stage iteration of forcing notions, and let $S = \{\alpha < \kappa : \mathrm{cf}(\alpha) = \omega_1\}$. For $\alpha < \kappa$ we will write $\Vdash_\alpha$ instead of $\Vdash_{\mathbf{P}_\alpha}$. If $G \subseteq \mathbf{P}_\kappa$ is generic over V, we will write G_α for $G \cap \mathbf{P}_\alpha$ and V_α for $V[G_\alpha]$.

We will say that $\mathbf{P}_\kappa$ is *innocuous* if

(1) For all $\alpha \in \kappa \setminus S$, $\mathbf{Q}_\alpha = \mathrm{Fn}(\omega, 2)$, and

(2) For all $\alpha \in S$, $\mathbf{Q}_\alpha$ has underlying set ω_1 and there is a function $f_\alpha : \omega_1 \to \alpha$, increasing, continuous and cofinal, as well as a club $C_\alpha \subseteq \omega_1 \cap$ *limits* such that for all $\delta \in C_\alpha$

$$\Vdash_\alpha \text{“}\dot{\mathbf{Q}}_\alpha \cap \delta \in V_{f_\alpha(\delta)}\text{”}, \quad \text{and} \quad \Vdash_\alpha \text{“}\dot{\mathbf{Q}}_\alpha \cap \delta \ll_{V_{f_\alpha(\delta)}} \dot{\mathbf{Q}}_\alpha\text{”}.$$

In the remainder of this section let $\langle\langle \mathbf{P}_\alpha \rangle_{\alpha \le \kappa}, \langle \dot{\mathbf{Q}} \rangle_{\alpha < \kappa}\rangle$ be an innocuous iteration.

3.2. Lemma [Ju 2.6]. *$\mathbf{P}_\kappa$ satisfies the c.c.c.*

3.3. Lemma [Ju 2.7]. *If θ is a sufficiently large regular cardinal, $N \prec H(\theta)$ and S, $\langle f_\alpha \rangle_{\alpha \in S}$, $\langle C_\alpha \rangle_{\alpha \in S}$, $\langle\langle \mathbf{P}_\alpha \rangle_{\alpha \le \kappa}$, $\langle \dot{Q} \rangle_{\alpha < \kappa}\rangle$ are all elements of N, then $\mathbf{P}_\kappa \cap N \ll_V \mathbf{P}_\kappa$.*

3.4. Corollary [Ju 2.5]. *$\mathbf{P}_\kappa$ is harmless.*

3.5. Lemma [Ju 2.9]. *If $\alpha \in \kappa \setminus S$, then*

$$\Vdash_\alpha \text{“}\mathbf{P}_\kappa / \mathbf{P}_\alpha \textit{ is isomorphic to an innocuous iteration”}.$$

3.6. Lemma [Ju 5.1(a)]. *Let $n \in \omega \setminus \{0\}$ and let $\phi(a_1, \ldots, a_k)$ be a Σ^1_n-formula with all parameters shown. If $\mathbf{P}_0$ and $\mathbf{P}_1$ are both everywhere uncountable harmless forcing notions and $a_1, \ldots, a_k$ are reals in the ground model, then*

$$\Vdash_{\mathbf{P}_0} \phi(\check{a}_1, \ldots, \check{a}_k) \Leftrightarrow \Vdash_{\mathbf{P}_1} \phi(\check{a}_1, \ldots, \check{a}_k).$$

($\mathbf{P}$ *is* everywhere uncountable *if* $\mathbf{P} \mid p$ *does not have a countable dense subset for any* $p \in \mathbf{P}$.)

§4. Preliminary lemmas

4.1. Lemma. *If $\rho : \mathfrak{B}_\kappa \to \Sigma_\kappa$ is a lifting for μ_κ then there is an $A \subseteq \kappa$ such that $|A| \le 2^{\aleph_0}$ and $\rho'' \mathfrak{B}_\kappa \subseteq \Sigma^\kappa_A$.*

PROOF. Using the fact that each member of Σ_κ factors through a countable set of coordinates, represent A as $\bigcup_{\alpha<\omega_1} A_\alpha$, where $|A_\alpha| \le 2^{\aleph_0}$ and $\rho''\mathcal{B}^\kappa_{A_\alpha} \subseteq \Sigma^\kappa_{A_{\alpha+1}}$ for each α. ■

4.2. COROLLARY. *If μ_κ has no projective lifting for $\kappa \le 2^{\aleph_0}$, then for every κ there is no projective lifting for μ_κ.* ■

4.3. LEMMA. *If $L \in \Sigma_{\kappa+\lambda}$ and $g \in 2^\lambda$ then $(\pi^{\kappa+\lambda}_\kappa)''(L \cap (2^\kappa \times \{g\})) \in \Sigma_\kappa$. Also, if $A, B \subseteq 2^\kappa$ and $A \times 2^\lambda$, $B \times 2^\lambda$ are separated by some set $L \in \Sigma_{\kappa+\lambda}$, then A, B are separated by $(\pi^{\kappa+\lambda}_\kappa)''(L \cap (2^\kappa \times \{g\})) \in \Sigma_\kappa$ for any $g \in 2^\lambda$.* ■

4.4. COROLLARY. *If $\rho : \mathcal{B} \to \Sigma^{\kappa+\lambda}_\kappa$ is a projective lifting for $\mu_{\kappa+\lambda}$ with respect to a subalgebra $\mathcal{B} \subseteq \mathcal{B}^{\kappa+\lambda}_\kappa$ and ρ can be extended to a projective lifting for $\mu_{\kappa+\lambda}$ with respect to the algebra generated by $\mathcal{B} \cup \{E\}$ for some $E \in \mathcal{B}^{\kappa+\lambda}_\kappa$, then there is such an extension $\bar\rho$ which satisfies $\bar\rho(E) \in \Sigma^{\kappa+\lambda}_\kappa$.*

PROOF. Extend ρ to a lifting for $\mu_{\kappa+\lambda}$ with respect to the algebra generated by $\mathcal{B} \cup \{E\}$ and denote this extension also by ρ. Then $\rho(E)$ separates A and B where

$$A = \{\rho(X) : X \in \mathcal{B},\ \mu_{\kappa+\lambda}(X \setminus E) = 0\},$$

$$B = \{\rho(X) : X \in \mathcal{B},\ \mu_{\kappa+\lambda}(X \cap E) = 0\}.$$

Since $\rho''\mathcal{B} \subseteq \Sigma^{\kappa+\lambda}_\kappa$, an application of 4.3 with $L = \rho(E)$ yields a projective set $S \in \Sigma^{\kappa+\lambda}_\kappa$ which is the section of $\rho(E)$ at some (any) $g \in 2^\lambda$ and such that $A \subseteq S$, $B \cap S = 0$. Since $\mu_{\kappa+\lambda}(E \Delta \rho(E)) = 0$, by Fubini's Theorem we may choose $g \in 2^\lambda$ so that $\mu_{\kappa+\lambda}(S \Delta E) = 0$. Take $\bar\rho(E) = S$. ■

The following properties of the functions $g^*, \bar g$ associated with a one-to-one function $g : \omega \to \lambda$ (see §2, (f)) will be useful:

4.5. LEMMA. (a) *g^* is continuous.*

(b) *$\bar g \mid \Sigma_\omega : \Sigma_\omega \to \Sigma^\lambda_{\mathrm{ran}(g)}$ is Boolean homomorphism which is measure-preserving with respect to the measures μ_ω on its domain and μ_λ on its range.* ■

§5. No projective liftings for the Haar measure on $\{0,1\}^\kappa$

5.1. THEOREM. $\mathrm{Con}(ZFC) \to \mathrm{Con}(ZFC + 2^{\aleph_0} = \aleph_2 + \forall\kappa\ \mu_\kappa$ *does not have a projective lifting*$)$.

PROOF. Assume $V = L$, and let $\kappa = \omega_2$. The desired model will be produced by a κ-stage innocuous forcing iteration. Hence $V_\kappa \Vdash 2^{\aleph_0} = \aleph_2$. By 4.2 we need only ensure that none of μ_ω, μ_{ω_1}, and μ_{ω_2} has a projective lifting in V_κ.

Let $\lambda \in \{\omega, \omega_1, \omega_2\}$. Since $\mathbf{P}_\kappa$ satisfies the c.c.c., if $\Vdash_\kappa$ "$\dot{A} \subseteq \check{\lambda}$ has size $\aleph_0$", then there is a $B \in [\lambda]^{\aleph_0}$ such that $\Vdash_\kappa$ "$\dot{A} \subseteq \check{B}$". It follows that if $\Vdash_\kappa$ "$\dot{S} \in \Sigma_\lambda$", then there are a one-to-one function g and a name $\dot{S}'$ such that

$$\Vdash_\kappa \text{“}\dot{S}' \in \Sigma^\lambda_{\operatorname{ran}(g)},\ \dot{S}' \in \Sigma_\omega, \text{ and } \dot{S} = \check{\bar{g}}(\dot{S}')\text{”}.$$

If $\Vdash_\kappa$ "$\dot{\rho} : \mathcal{B}_\lambda \to \Sigma_\lambda$ is a projective lifting for μ_λ", then for each $\dot{c}$ such that

$$\Vdash_\kappa \text{“}\dot{c} \in \mathcal{C}_\omega \text{ codes an element of } \mathcal{B}_\omega\text{”},$$

and for each one-to-one $g : \omega \to \lambda$ we can find a one-to-one $h : \omega \to \lambda$ and a name $\dot{d}$ such that

$$\Vdash_\kappa \text{“}\dot{d} \in \mathcal{C}_\omega \text{ and } \dot{\rho}(\bar{g}(\operatorname{eval}(\dot{c}))) = \bar{h}(\operatorname{eval}(\dot{d}))\text{”}.$$

Pick one such pair $\langle \dot{d}, h\rangle$ for each pair $\langle \dot{c}, g\rangle$ and write $\dot{\rho}(\langle \dot{c}, g\rangle) = \langle \dot{d}, h\rangle$. Take the names $\dot{c}$, $\dot{d}$ to be nice names of the form $\bigcup_{n\in\omega} \{\check{n}\} \times A_n$, where $A_n \subseteq \mathbf{P}_\kappa$ is a countable antichain.

For $\alpha < \kappa$, write $\underline{\alpha} = \min\{\alpha, \lambda\}$, and $\dot{\mathcal{C}}_\alpha = \{\langle \dot{c}, g\rangle : \dot{c}$ is a nice $\mathbf{P}_\alpha$-name for a code of a projective subset of 2^ω and $g : \omega \to \underline{\alpha}$ is one-to-one$\}$. By *CH*, $|\dot{\mathcal{C}}_\alpha| \le \aleph_1$ for every $\alpha < \kappa$.

Let $C_{\dot{\rho}}$ denote a club $\subseteq \kappa$ such that $\dot{\rho}''\dot{\mathcal{C}}_\alpha \subseteq \dot{\mathcal{C}}_\alpha$ whenever $\alpha \in C_{\dot{\rho}}$ and $\operatorname{cf}(\alpha) = \omega_1$.

Let S_ω, S_{ω_1}, S_{ω_2} be three pairwise disjoint stationary subsets of $\{\alpha < \kappa : \operatorname{cf}(\alpha) = \omega_1\}$. For each $\lambda \in \{\omega, \omega_1, \omega_2\}$, fix a $\diamondsuit_{\omega_2}(S_\lambda)$-sequence $\langle \dot{\rho}_\alpha : \alpha \in S_\lambda\rangle$ where $\dot{\rho}_\alpha : \dot{\mathcal{C}}_\alpha \to \dot{\mathcal{C}}_\alpha$.

We will construct $\mathbf{P}_\kappa$ in such a way that when $\alpha \in S_\lambda$ and

$$\Vdash_\alpha \text{“}\dot{\rho}_\alpha \text{ is a projective lifting for } \mu_{\underline{\alpha}}\text{”},$$

then $\mathbf{Q}_\alpha$ adds an open Baire set $\dot{X}_\alpha \subseteq 2^{\underline{\alpha}}$ such that

(*) $\Vdash_{\mathbf{P}_{\alpha+1}\times \mathrm{Fn}(\omega_1,2)}$ "$\dot{\rho}_\alpha$ cannot be extended to a lifting for $\mu_{\underline{\alpha}}$ with respect to any subalgebra of $\mathcal{B}_{\underline{\alpha}}$ which includes $(\mathcal{B}_{\underline{\alpha}})^{V_\alpha} \cup \{\dot{X}_\alpha\}$".

We now sketch a proof that such an iteration indeed yields 5.1. Suppose not and fix $\lambda, \dot{\rho}$ and p such that $p \Vdash_\kappa$ "$\dot{\rho}$ is a projective lifting for μ_λ". Pick $\alpha \in S_\lambda \cap C_{\dot{\rho}}$ such that $\dot{\rho} \restriction \dot{\mathcal{C}}_\lambda = \dot{\rho}_\lambda$.

5.2. Claim. $\Vdash_\alpha$ "$\dot{\rho}_\alpha$ is a projective lifting for $\mu_{\underline{\alpha}}$".

Proof. Notice that every pair $\langle\langle \dot{c}, g\rangle, \dot{\rho}_\alpha(\langle \dot{c}, g\rangle)\rangle$ is already present at some stage $\beta < \alpha$ such that $\operatorname{cf}(\beta) \neq \omega_1$. In V_β, both $\mathbf{P}_\alpha/\mathbf{P}_\beta$ and $\mathbf{P}_\kappa/\mathbf{P}_\beta$ are everywhere uncountable harmless forcing notions by 3.5. Now the claim follows from the observation that the property of being a projective lifting can be expressed as a conjunction of uncountably many formulas (of the form '$\rho(2^{\underline{\alpha}}\backslash E) = 2^{\underline{\alpha}}\backslash\rho(E)$', '$\rho(E \cup F) = \rho(E) \cup \rho(F)$', '$\mu_{\underline{\alpha}}(E\Delta\rho(E)) = 0$', '$\mu_{\underline{\alpha}}(E\Delta F) = 0 \Rightarrow \rho(E) = \rho(F)$' for Baire sets E,F), every one of which is, by 3.6, absolute between V_α and V_κ. A more detailed proof can be found in [Ju §6]. ■

It follows that at stage α of the construction we chose $\mathbf{Q}_\alpha$ and $\dot{X}_\alpha$ so that (*) holds. Since $\dot{\rho}$ is a name for a lifting, there is a $\mathbf{P}_\kappa$-name $\dot{Y}$ such that $p \Vdash_\kappa$ "$\dot{\rho}(\dot{X}_\alpha) = \dot{Y}$". By 4.4, this means there is a P_κ-name $\dot{d}$ for a code of a projective set and a one-to-one $g : \omega \to \underline{\alpha}$ such that

(**) $p \Vdash_\kappa$ "$\dot{\rho}_\alpha$ can be extended to a projective lifting for $\mu_{\underline{\alpha}}$ with respect to the subalgebra of $\mathcal{B}_{\underline{\alpha}}$ generated by $(\mathcal{B}_{\underline{\alpha}})^{V_\alpha} \cup \{\dot{X}_\alpha\}$ by letting $\dot{\rho}_\alpha(\dot{X}_\alpha) = \check{\bar{g}}(\mathrm{eval}(\dot{d}))$".

Since $\mathbf{P}_\kappa/\mathbf{P}_{\alpha+1}$ is everywhere uncountable and harmless (i.e., this is true in $V_{\alpha+1}$), and since the property of being a projective lifting is absolute as in the proof of 5.2, (**) contradicts (*). (This argument is given in gross detail in [Ju §6].)

Now fix $\lambda \in \{\omega, \omega_1, \omega_2\}$ and $\alpha \in S_\lambda$ so that $\Vdash_\alpha$ "$\dot{\rho}_\alpha$ is a projective lifting for $\mu_{\underline{\alpha}}$". In the remainder of this paper we will show how to choose $\dot{\mathbf{Q}}_\alpha, \dot{X}_\alpha$ such that (*) holds.

Let G_α be a generic subset of $\mathbf{P}_\alpha$. Until further notice we work in $V_\alpha = V[G_\alpha]$. Let $\rho = \mathrm{val}(\dot{\rho}_\alpha, G_\alpha)$. Then ρ is a projective lifting for $\mu_{\underline{\alpha}}$. In V find a countable infinite $D \subseteq \underline{\alpha}$ such that $\rho([s]_{\underline{\alpha}}) \in \Sigma^{\underline{\alpha}}_D$ for each $s \in \mathrm{Fn}(D,2)$. Fix in V an ordering of D into type ω and let $<$ denote the induced lexicographical ordering on 2^D. Call the eventually constant sequences in 2^D *rational numbers*.

For $x \in 2^D$, define $\tau x \in 2^{\underline{\alpha}}$ as follows: $\tau x(\xi) = x(\xi)$ if $\xi \in D$, $\tau x(\xi) = 0$ if $\xi \in \underline{\alpha}\backslash D$. For $A \subseteq 2^D$, let $\tau A = A \times 2^{\underline{\alpha}\backslash D}$.

5.3. Main Lemma. *There exists a continuous increasing function $f : \omega_1 \to \alpha$ and there is a club $C \subseteq \omega_1 \cap$ limits, and for every $\eta < \omega_1$ there is a countable forcing notion $\mathbf{Q}(\eta)$ with underlying set $\omega \cdot \eta$ so that $\mathbf{Q} = \bigcup_{\eta<\omega_1} \mathbf{Q}(\eta)$ adds an open Baire set $\dot{X} \subseteq 2^{\underline{\alpha}}$ and the following conditions hold*:

(i) $\forall \eta \in C \cup (\omega_1 \cap$ *successors*$)$, $\mathbf{Q}(\eta) \in V_{f(\eta)}$ *and* $\mathbf{Q}(\eta) \ll_{V_{f(\eta)}} \mathbf{Q}$.

(ii) *For each* $\mathbf{Q} \times \mathrm{Fn}(\omega_1,2)$-*name χ for a code of a projective subset of $2^{\underline{\alpha}}$ and for each $(p,r) \in \mathbf{Q} \times \mathrm{Fn}(\omega_1,2)$, there is a cofinal set $T \subseteq \omega_1$ so that for all $\eta \in T$ there are an open Baire set $U \subseteq 2^{\underline{\alpha}}$, an $x \in 2^{\underline{\alpha}}$ and a condition $(p',r') \leq (p,r)$ in* $\mathbf{Q} \times \mathrm{Fn}(\omega_1,2)$ *such that*

(a) $x, U, \rho(U) \in V_{f(\eta+1)}$, *and*

(b) *either* $(p',r') \Vdash_{\mathbf{Q}(\eta+1)\times\mathrm{Fn}(\omega_1,2)}$ "$\check{x} \in \mathrm{eval}(\chi) \cap \check{\rho}(\check{U})$ *and* $\check{U} \cap \dot{X} = \varnothing$"
or $(p',r') \Vdash_{\mathbf{Q}(\eta+1)\times\mathrm{Fn}(\omega_1,2)}$ "$\check{x} \in \check{\rho}(\check{U})\backslash\mathrm{eval}(\chi)$ *and* $\check{U} \subseteq \dot{X}$".

The reader will certainly have guessed at this point that in our construction of $\mathbf{P}_\kappa$ we will take $\mathbf{Q}$ from the Main Lemma for $\mathbf{Q}_\alpha$, a club $C_\alpha \subseteq C$ such that $C_\alpha \in V$ and $\omega \cdot \delta = \delta$ for every $\delta \in C_\alpha$, and an $f_\alpha \in V$ which agrees on C_α with f to witness innocuousness. Now (*) is almost obvious, except that one has to make sure that "$\Vdash^{V_\alpha}_{\mathbf{Q}(\eta+1)\times \mathrm{Fn}(\omega_1,2)}$" can be replaced by "$\Vdash^{V_\alpha}_{\mathbf{Q}\times \mathrm{Fn}(\omega_1,2)}$". This can be verified in two stages: first notice that by 3.5 and 3.6 the relation "$\Vdash^{V_\alpha}_{\mathbf{Q}(\eta+1)\times \mathrm{Fn}(\omega_1,2)}$" can be replaced by "$\Vdash^{V_{f(\eta+1)}}_{\mathbf{Q}(\eta+1)\times \mathrm{Fn}(\omega_1,2)}$". In $V_{f(\eta+1)}$, $\mathbf{Q}$ looks like a two-stage iteration $\mathbf{Q}(\eta+1) * \dot{\mathbf{R}}$, where $\dot{\mathbf{R}}$ is harmless. It follows now from 3.6 and the fact that only a countable fragment of $\mathrm{Fn}(\omega_1,2)$ is needed to compute $\mathrm{eval}(\chi)$ that "$\Vdash^{V_{f(\eta+1)}}_{\mathbf{Q}(\eta+1)\times \mathrm{Fn}(\omega_1,2)}$" can be replaced by "$\Vdash^{V_{f(\eta+1)}}_{\mathbf{P}_\alpha/\mathbf{P}_{f(\eta+1)} * \mathbf{Q}\times \mathrm{Fn}(\omega_1,2)}$", which is exactly what we need for (*). This reasoning is given in greater detail in [Ju §§5,6].

Proof of 5.3. This proof follows closely the argument in [S].

For $a \neq b$, $a,b \in 2^D$, let

$$(a,b) = \begin{cases} \{x \in 2^D : a < x < b\}, & \text{if } a < b, \\ \{x \in 2^D : b < x < a\}, & \text{if } b < a. \end{cases}$$

Let $\mathcal{S} = \{a \in (2^D)^{\omega+1} : a \restriction \omega$ is a monotone sequence of rational numbers such that $(a_n, a_{n+1}) \neq \emptyset$ for all n, and $a_\omega = \lim_{n\to\infty} a_n$ is not rational$\}$.

Given a sequence $\langle a^\xi : \xi < \eta \leq \omega_1 \rangle$ of elements of $\mathcal{S}$ with pairwise distinct a^ξ_ω's, define a partial order $\mathbf{Q}(\langle a^\xi : \xi < \eta \rangle)$ as follows:

$$p \in \mathbf{Q} \text{ iff } p = (U_p, h_p), \qquad \text{where}$$

(i) $U_p \subseteq 2^D$ is open, $\mu_D(\bar{U}_p) < 1/2$ (the bar denotes closure in the usual topology on 2^D);

(ii) $h_p : U_p \to 2$;

(iii) There are $n^* \in \omega$, b_l $(l \leq n^*)$, I_l $(l < n^*)$ such that $0 = b_0 < b_1 < \cdots < b_{n^*-1} < b^*_n = 1$ and $\bar{I}_l \subseteq (b_l, b_{l+1})$ and for each $l < n^*$ either

(a) I_l is a rational interval and $h_p \restriction I_p$ is constant, or

(b) $(\exists \xi < \xi_0)(\exists n(l) < \omega)\ I_l = \bigcup_{m=n(l)}^\infty (a^\xi_{2m}, a^\xi_{2m+1})$ and $h_p \restriction (a^\xi_{4m+2k}, a^\xi_{4m+2k+1}) \equiv k$, for $k \in \{0,1\}$ and $n(l) \leq 2m + k < \omega$.

The partial order is defined by: $p \leq q \Leftrightarrow U_p \supseteq U_q$, $h_p \supseteq h_q$, $U_p \cap \bar{U}_q = U_q$.

The $\mathbf{Q}$ we are going to construct will be isomorphic to $\mathbf{Q}(\langle a^\xi : \xi < \omega_1 \rangle)$ for some sequence of a^ξ's to be chosen. $\dot{X}$ will be a $\mathbf{Q}$-name for the open set $\tau \dot{Y}$, where

$$\Vdash_{\mathbf{Q}} \text{“}\dot{Y} = \bigcup \Big\{ (a,b) : (a,b) \subseteq 2^D \text{ is a rational interval and } \exists p \in \dot{G}_{\mathbf{Q}} \text{ such that } (a,b) \subseteq U_p \text{ and } h_p \restriction (a,b) \equiv 0 \Big\}\text{”}.$$

($\dot{G}_{\mathbf{Q}}$ is the canonical name for the generic filter over $\mathbf{Q}$.)

$\mathbf{Q}(\eta)$ will be a forcing notion with underlying set $\omega \cdot \eta$ and isomorphic to

$\mathbf{Q}(\langle a^\xi : \xi < \eta\rangle)$. This is a purely technical requirement to bring our construction into line with the general framework of innocuous iterations. For all practical purposes we may think of $\mathbf{Q}(\eta)$ as being $\mathbf{Q}(\langle a^\xi : \xi < \eta\rangle)$. However, in order to meet requirement (i) of the Main Lemma, we need to be sure that if $\mathbf{Q}(\langle a^\xi : \xi < \eta\rangle) \in V_\beta$ for some $\beta < \alpha$, then also $\mathbf{Q}(\eta) \in V_\beta$. This can be easily arranged by choosing an identification of $\mathbf{Q}(\eta+1)\backslash\mathbf{Q}(\eta)$ with $[\omega\cdot\eta, \omega\cdot(\eta+1))$ at the same time and in the same intermediate model as a^η. From now on we identify $\mathbf{Q}(\eta)$ and $\mathbf{Q}(\langle a^\xi : \xi < \eta\rangle)$.

Fix a function

$$F : \omega_1\backslash\{0\} \xrightarrow{\text{onto}} \omega_1 \times \omega_1 \times \omega_1$$

so that if $F(\xi) = \langle \eta_0, \eta_1, \zeta\rangle$, then $\eta_0 < \xi$.

Let $\langle \chi_\xi(\eta) : \xi < \omega_1\rangle$ list all pairs $\langle \dot{c}, g\rangle$ where $\dot{c}$ is a $\mathbf{Q}(\eta) \times \mathrm{Fn}(\omega_1,2)$-name for an element of $\mathcal{C}_\omega$ and $g : \omega \to \underline{\alpha}$ is one-to-one. Also list $\mathbf{Q}(\eta) \times \mathrm{Fn}(\omega_1,2)$ as $\langle s_\zeta(\eta) : \zeta < \omega_1\rangle$.

It remains to construct the sequence $\langle a^\xi : \xi < \omega_1\rangle$, the club C, and the function f.

For every rational interval $(a,b) \subseteq 2^D$, we pick $g_{(a,b)}$, $h_{(a,b)}$, $c_{(a,b)}$ and a $\mathbf{P}_\alpha$-name $\dot{d}_{(a,b)}$ such that $\tau(a,b) = \bar{g}_{(a,b)}(\mathrm{eval}(c_{(a,b)}))$ and $\dot{\rho}_\alpha(\langle \check{c}_{(a,b)}, g_{(a,b)}\rangle) = \langle \dot{d}_{(a,b)}, h_{(a,b)}\rangle$.

Pick a function $f' : \omega_1 \to \alpha$, continuous, increasing and cofinal in α, and choose $f(0) \geq f'(0)$ large enough so that $\dot{d}_{(a,b)}$ is a $\mathbf{P}_{f'(0)}$-name for every rational interval $(a,b) \subseteq 2^D$.

Limit stages take care of themselves.

Now suppose $\mathbf{Q}(\eta)$ and $f(\eta)$ are given. Let $F(\eta) = \langle \eta_0, \eta_1, \zeta\rangle$ and let $\chi = \chi_{\eta_1}(\eta_0)$. If $\chi = \langle \dot{c}, g\rangle$ is a $\mathbf{Q}(\eta) \times \mathrm{Fn}(\omega_1,2)$-name for a member of $\Sigma_D^{\underline{\alpha}}$, then we shall write $\mathrm{eval}(\chi)$ instead of $\bar{g}(\mathrm{eval}(\dot{c}))$. Let $(p^*, r^*) = s_\zeta(\eta_0) \in \mathbf{Q}(\eta_0) \times \mathrm{Fn}(\omega_1,2)$. Choose an ordinal $\gamma_\eta \geq f'(\eta+1), f(\eta)$ such that $\chi, \mathbf{Q}(\eta) \in V_{\gamma_\eta}$.

5.4. Claim. *There is an* $a^\eta \in \mathcal{S}$ *such that*

(i) $\mathbf{Q}(\eta) \ll_{V_{\gamma_\eta}} \mathbf{Q}(\eta+1)$,

(ii) $\exists (p', r') \leq (p^*, r^*)$, $(p', r') \in \mathbf{Q}(\eta+1) \times \mathrm{Fn}(\omega_1,2)$, $\exists k \in \omega$ *such that*

either $(p', r') \Vdash_{\mathbf{Q}(\eta+1)\times\mathrm{Fn}(\omega_1,2)}$ "$\tau a^\eta_\omega \in \mathrm{eval}(\chi) \cap \check{\rho}(\bigcup_{m\geq k} \tau(a^\eta_{4m+2}, a^\eta_{4m+3}))$ *and* $\dot{X} \cap \bigcup_{m\geq k} \tau(a^\eta_{4m+2}, a^\eta_{4m+3}) = \varnothing$"

or $(p', r') \Vdash_{\mathbf{Q}(\eta+1)\times\mathrm{Fn}(\omega_1,2)}$ "$\tau a^\eta_\omega \in \check{\rho}(\bigcup_{m\geq k} \tau(a^\eta_{4m}, a^\eta_{4m+1}))\backslash\mathrm{eval}(\chi)$ *and* $\bigcup_{m\geq k} \tau(a^\eta_{4m}, a^\eta_{4m+1}) \subseteq \dot{X}$".

Assuming 5.4, the Main Lemma is proven as follows: pick a^η as in 5.4 and let $f(\eta+1) \geq \gamma_\eta$ be large enough so that $\mathbf{Q}(\eta+1) \in V_{f(\eta+1)}$ and $\mathrm{cf}(f(\eta+1)) = \omega$.

Finally, we have to choose C. Notice that our construction yields automatically that $\mathbf{Q}(\eta) \ll_{V_{f(\eta)}} \mathbf{Q}$ for all η. What might fail is the condition $\mathbf{Q}(\eta) \in V_{f(\eta)}$ for some limit ordinal η. However, $\mathbf{Q}$ has a $\mathbf{P}_\alpha$-name $\dot{\mathbf{Q}}$. since $\mathbf{P}_\alpha$ satisfies the c.c.c., a standard closure argument yields a club $C \subseteq \omega_1$ such that $\mathbf{Q} \cap \omega \cdot \eta \in V_{f(\eta)}$ for all $\eta \in C$. We choose such a C, and this completes the proof of the Main Lemma.

Proof of 5.4. Write γ instead of γ_η. Let $K \subseteq 2^D$ be a homeomorphic copy of the Cantor set (i.e., $K \approx 2^\omega$) such that $\mu_D(K) \geq 1/2$, every open subset of 2^D is either disjoint from K or intersects K on a set of positive measure (i.e., K is *self-supporting*), and

$$\tau K \cap \left(\bigcup \{\rho(\tau(a,b))\backslash \tau(a,b) : (a,b) \subset 2^D \text{ is a rational interval}\} \cap \tau \bar{U}_p = \varnothing.$$

By definition of D, we have $\rho(\tau(a,b)) \in \Sigma_D^\alpha$ for each rational interval $(a,b) \subseteq 2^D$. Since each $\rho(\tau(a,b))$ is coded in $V_{f(0)} \subseteq V_\gamma$, we may choose $K \in V_\gamma$.

Let $a_\omega^\eta \in K$ be the image under some homeomorphism $2^\omega \to K$ coded in V_γ of the Cohen real added by $\mathbf{Q}_{\gamma+1} = \mathrm{Fn}(\omega, 2)$. Either $\tau a_\omega^\eta \in \rho(\tau(0, a_\omega^\eta))$, or $\tau a_\omega^\eta \in \rho(\tau(a_\omega^\eta, 1))$, without loss of generality of the former.

In $V_{\gamma+2}$, let $\langle b_n : n \in \omega\rangle$ be a sequence of rational numbers in 2^D increasing to a_ω^η such that $(b_n, b_{n+1}) \neq \varnothing$ for all n. In $V_{\gamma+3}$, define $\langle n(l) : l \in \omega\rangle$ by

$$n(0) = 0, \quad n(l+1) = n(l) + e(l), \quad l \in \omega,$$

where $e \in (\omega \backslash \{0\})^\omega$ is the Cohen real added by $\mathbf{Q}_{\gamma+2} \cong \mathrm{Fn}(\omega, \omega\backslash\{0\})$.

5.5. Remarks. (a) $\forall b \in (0, a_\omega^\eta)$, $\tau a_\omega^\eta \in \rho(\tau(b, a_\omega^\eta))$. (If not, choose a rational number $x \in (s, a_\omega^\eta)$. Then $\tau a_\omega^\eta \in \rho(\tau(0,x)) \backslash \tau(0,x)$, contradicting the definition of K.)

(b) For $m < 4$, $k < \omega$, let $A_m^k = \bigcup_{l \geq k} (b_{n(4l+m)}, b_{n(4l+m+1)})$. Then $\{A_m^0 : m < 4\}$ is a partition of (b_0, a_ω^η). Hence $a_\omega^\eta \in \rho(\tau A_{\bar{m}}^0)$ for some unique $\bar{m} < 4$. From (a) it follows that $\tau a_\omega^\eta \in \rho\tau(A_{\bar{m}}^k)$ for all k.

(c) Suppose we would rather have $\tau a_\omega^\eta \in \rho(\tau A_{\bar{\bar{m}}}^0)$ for some $\bar{\bar{m}} \neq \bar{m}$. It is an easy exercise to show that this can be accomplished by replacing e by an equivalent Cohen real $\bar{e}$, which satisfies $\bar{e}(i) = e(i + i_0)$ for some fixed $i_0 \in \omega$ and for all sufficiently large i, and furthermore we can arrange for any given k that $\bar{e}|k = e|k$.

Now we return to the construction of a^η. First we show that 5.4(ii) will hold for some suitable modification of e as indicated in 5.5(c), if we let $a_l^\eta = b_{n(l)}$ for $l < \omega$. Tentatively let $a_l^\eta = b_{n(l)}$, $l < \omega$, with the $n(l)$'s and $\mathbf{Q}(\eta+1)$ defined using the unmodified e. If 5.4(ii) fails we can modify e to make it hold, as follows:

Let $p_1 = (U_{p^*} \cup A_0^k \cup A_2^k,\ h_{p^*} \cup 0_{A_0^k} \cup 1_{A_2^k})$, where i_X, for $i = 0, 1$, is the function with domain X and constant value i, and k is large enough so that p_1 is a condition, i.e., $\bar{U}_{p^*} \cap [b_{n(4k)}, a^\eta_\omega] = \emptyset$ and $\mu_D(\bar{U}_{p^*}) < 1/2$. Choose $(p',r') \leq (p_1,r^*) \leq (p^*,r^*)$, $(p',r') \in \mathbf{Q}(\eta+1) \times \mathrm{Fn}(\omega_1,2)$, so that (p',r') decides "$\tau\check{a}^\eta_\omega \in \mathrm{eval}(\chi)$". (If $\chi = \langle \dot{c}, g\rangle$ but $\dot{c}$ is not a $\mathbf{Q}(\eta+1) \times \mathrm{Fn}(\omega_1,2)$-name for a code of a projective set, then interpret $\mathrm{eval}(\chi)$ as the empty set.)

If $(p',r') \Vdash_{\mathbf{Q}(\eta+1)\times\mathrm{Fn}(\omega_1,2)}$ "$\tau\check{a}^\eta_\omega \in \mathrm{eval}(\chi)$", then

$$V_\alpha \Vdash \text{“}(p',r') \Vdash_{\mathbf{Q}(\eta+1)\times\mathrm{Fn}(\omega_1,2)} \text{“}\tau\check{a}^\eta_\omega \in \mathrm{eval}(\chi)\text{””}. \tag{1}$$

Now $\mathrm{dom}(r') \subseteq \beta$ for some infinite $\beta < \omega_1$. So (1) can be rewritten

$$V_\alpha \Vdash \text{“}(p',r') \Vdash_{\mathbf{Q}(\eta+1)\times\mathrm{Fn}(\beta,2)} \text{“}\tau\check{a}^\eta_\omega \in \mathrm{eval}(\chi) \text{ holds in an extension by an everywhere uncountable harmless forcing notion””}. \tag{2}$$

Since $\mathbf{Q}(\eta+1) \times \mathrm{Fn}(\beta,2) \in V_{\gamma+3}$, and since, by 3.5, V_α is an extension of $V_{\gamma+3}$ by an everywhere uncountable harmless forcing notion, (2) entails

$$V_{\gamma+3} \Vdash \text{“}(p',r') \Vdash_{\mathbf{Q}(\eta+1)\times\mathrm{Fn}(\beta,2)} \text{“}\tau\check{a}^\eta_\omega \in \mathrm{eval}(\chi) \text{ holds in an extension by an everywhere uncountable harmless forcing notion””}. \tag{3}$$

By 3.6,

$$V_{\gamma+3} \Vdash \text{“}(p',r') \Vdash_{\mathbf{Q}(\eta+1)\times\mathrm{Fn}(\omega_1,2)} \text{“}\tau\check{a}^\eta_\omega \in \mathrm{eval}(\chi)\text{””}. \tag{4}$$

Now (4) means that for some $e_0 \in \mathrm{Fn}(\omega, \omega\setminus\{0\})$,

$$V_{\gamma+2} \Vdash \text{“}e_0 \Vdash_{\mathbf{Q}_{\gamma+2}} \text{“}(\dot{p}',r') \Vdash_{\mathbf{Q}(\eta+1)\times\mathrm{Fn}(\omega_1,2)} \text{“}\tau\check{a}^\eta_\omega \in \mathrm{eval}(\chi)\text{”””}. \tag{5}$$

By 5.5(c), we can replace e by an equivalent Cohen real (over $V_{\gamma+2}$) which we shall also denote by e, maintaining $e_0 \subseteq e$, and so that $\tau a^\eta_\omega \in \rho(\tau A^k_2)$. Now $(p',r'),\mathbf{Q}(\eta+1)$ satisfy the first clause of 5.4(ii).

In the case where $(p',r') \Vdash_{\mathbf{Q}(\eta+1)\times\mathrm{Fn}(\omega_1,2)}$ "$\tau\check{a}^\eta_\omega \notin \mathrm{eval}(\chi)$", proceed in a similar fashion to modify e so that the second clause of 5.4(ii) will hold.

From now on, e (and a^η, and $\mathbf{Q}(\eta+1)$) are fixed. There remains to prove 5.4(i). We shall prove a somewhat stronger statement:

(i)$^+$ If V^+ is a model of a sufficiently powerful fragment of *ZFC* such that $\mathbf{Q}(\eta)$, $K \in V^+$, a^η_ω corresponds to a Cohen real over V^+, and e is $\mathrm{Fn}(\omega,\omega\setminus\{0\})$-generic over $V^+[a^\eta_\omega]$, then $\mathbf{Q}(\eta) \ll_{V^+} \mathbf{Q}(\eta+1)$.

In (i)$^+$, we do not require that V^+ is a subset of V_α.

Let $H \subseteq \mathbf{Q}(\eta)$ be predense, $H \in V^+$, and fix $p \in \mathbf{Q}(\eta+1)$. We must show that p is compatible with some member of H. Without loss of generality $p \notin \mathbf{Q}(\eta)$, so there are $q \in \mathbf{Q}(\eta)$, $l(0) \in \omega$, and rational numbers c_0, c_1 such that $0 < c_0 < a^\eta_\omega < c_1 < 1$, $b_{n(4l(0))-1} < c_0 < b_{n(4l(0))}$, $\bar{U}_q \cap [c_0, c_1] = \varnothing$, $U_p = U_q \cup A_0^{l(0)} \cup A_2^{l(0)}$, and $h_p = h_q \cup 0_{A_0^{l(0)}} \cup 1_{A_2^{l(0)}}$.

5.6. SUBCLAIM. *If $r \in \mathbf{Q}(\eta)$, $\tilde{H} \subseteq \mathbf{Q}(\eta)$ is* dense, *and $(b_0, b_1) \subseteq 2^D$ is an open interval disjoint from U_r, then $J = (d_0, d_1) \cap K \cap \bigcap\{\bar{U}_{r_1} : r_1 \in \tilde{H},\ r_1 \leq r\}$ is nowhere dense in $(d_0, d_1) \cap K$.*

PROOF. Since J is closed in $(d_0, d_1) \cap K$, if it is somewhere dense, then there is an interval $(a_0, a_1) \subseteq (d_0, d_1)$ such that $\varnothing \neq (a_0, a_1) \cap K \subseteq J$. Since K is self-supporting, $\mu_D((a_0, a_1) \cap K) = \epsilon > 0$. Take $r_2 \leq r$ such that $U_{r_2} \cap (a_0, a_1) = \varnothing$, and $\mu_D(U_{r_2}) > (1/2) - (\epsilon/2)$. Since $\tilde{H}$ is dense, there is an $r_1 \in \tilde{H}$, $r_1 \leq r_2$. Then $\mu_D(\bar{U}_{r_1}) \cap (a_0, a_1) < \epsilon/2$, and hence $\mu_D(A) > \epsilon/2$, where $A = (a_0, a_1) \cap K \setminus \bar{U}_{r_1}$. ■

Returning to the proof of (i)$^+$, let $\tilde{H} = \{r \in \mathbf{Q}(\eta) : (\exists q \in H) r \leq q_1\}$. It suffices to show that p is compatible with some $r \in \tilde{H}$. For each $k > \bar{n}(4l(0))$ let

$$T_k = \{t \in \mathbf{Q}(\eta) : U_t \text{ is the union of finitely many intervals whose endpoints are from } \{b_l : \bar{n}(4l(0)) \leq l \leq k\} \text{ and } \mu_D(U_q \cup U_t) < 1/2\}.$$

So T_k is finite and $q \cup t \in \mathbf{Q}(\eta)$, $q \cup t \leq q$ and $a^\eta_\omega \notin \bar{U}_t$, for each $t \in T_k$. For each k and $t \in T_k \setminus \bigcup_{l<k} T_l$ define

$$J_t = (b_k, c_1) \cap K \cap \bigcap\{\bar{U}_{r_1} : r_1 \in \tilde{H} \ \&\ r_1 \leq q \cup t\}.$$

Since $\tilde{H}$ is dense, the subclaim assures us that J_t is nowhere dense in $(b_k, c_1) \cap K$. By the choice of V^+ we have $K, \mathbf{Q}(\eta) \in V^+$, and hence also $\tilde{H}, J_t \in V^+$. Since a^η_ω corresponds to a Cohen real over V^+, we have $a^\eta_\omega \notin J_t$. It follows that for each $k > \bar{n}(4l(0))$ and each $t \in T_k \setminus \bigcup_{l<k} T_l$, there is an $r_t \in \tilde{H}$ such that $r_t \leq q \cup t$ and $a^\eta_\omega \notin \bar{U}_{r_t}$.

In $V^+[a^\eta_\omega]$, there are functions $g : \bigcup\{T_k : k > \bar{n}(4l(0))\} \to \omega$ and $G : \omega \to \omega$ such that $[b_{g(t)}, a^\eta_\omega] \cap \bar{U}_{r_t} = \varnothing$, $\mu_D(b_{g(t)}, a^\eta_\omega) < (1/2) - \mu(U_{r_t})$, $G(k) = \max\{g(t) : t \in T_k\}$.

Since e is $\mathrm{Fn}(\omega, \omega \setminus \{0\})$-generic over $V^+[a^\eta_\omega]$, there are arbitrarily large l for which $G(\bar{n}(4l+1)) < \bar{n}(4l+1) + e(4l+1) = \bar{n}(4l+2)$. Choose such an $l \geq l(0)$. Let $k = \bar{n}(4l+1)$, $t = (U_t, h_t)$, where $U_t = U_p \cap [b_{\bar{n}(4l(0))}, b_k]$, $h_t = h_p \mid U_t$. Then $t \in T_k \setminus \bigcup_{l<k} T_l$ and we have $r_t \in \tilde{H}$, $r_t \leq q \cup t$. Also, $[b_{G(k)}, a^\eta_\omega] \cap \bar{U}_{r_t} = \varnothing$ and hence $[b_{\bar{n}(4l+2)}, a^\eta_\omega] \cap \bar{U}_{r_t} = \varnothing$. Thus p and r_t are compatible, and this proves (i)$^+$. We have now proven 5.4, 5.3 and 5.1. ■ ■ ■

5.7. REMARK. The proof of 5.1 shows in fact that if κ is any regular cardinal not the successor of a cardinal of countable cofinality, then

$$\mathrm{Con}(ZFC) \Rightarrow \mathrm{Con}(ZFC + 2^{\aleph_0} = \kappa + (\forall\lambda)\ (\mu_\lambda \text{ has no projective lifting})).$$

The definition of innocuous iterations of length $>\aleph_2$ contains an additional condition (see [Ju, 2.3(b3)]) which is omitted here for the sake of making the argument more transparent. However, the reader familiar with [Ju] will have no difficulty in verifying that the statement (i)$^+$ in the proof of 5.4 implies that our iteration will satisfy this extra condition.

A similar and somewhat easier argument gives

$$\mathrm{Con}(ZFC) \Rightarrow \mathrm{Con}(ZFC + (\forall\lambda)\ (\text{there is no projective lifting for the algebra of Baire subsets of } 2^\lambda \text{ modulo the ideal of meager sets})).$$

The options for the size of the continuum are the same as before.

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